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Asymptotic optimality of multicenter Voronoi configurations for random field estimation

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Abstract— **This paper deals with multi-agent networks performing optimal estimation tasks. Consider a network of mobile agents with sensors that can take measurements of a spatial process in an environment of interest. Using the measurements, one can construct a kriging interpolation of the spatial field over the whole environment, with an associated prediction error at each point. We study the continuity properties of the prediction error, and consider as global objective functions the maximum prediction error and the generalized prediction variance. We study the network configurations that give rise to optimal field interpolations. Specifically, we show how, as the correlation between any two different locations vanishes, circumcenter and incenter Voronoi configurations become network configurations that optimize the maximum prediction error and the generalized prediction variance, respectively. The technical approach draws on tools from geostatistics, computational geometry, linear algebra, and dynamical systems.**

I. Introduction

Problem statement: Mobile sensor networks are envisioned to perform distributed sensing and data fusion tasks in a wide range of scenarios, including environmental monitoring, oceanographic research, and surveillance of critical infrastructures. This paper considers mobile sensor networks performing optimal estimation of physical processes modeled as spatial random fields. Standard interpolation techniques produce estimates of the spatial field at each point of the environment of interest. When a measure of the accuracy of the estimate is available, a natural objective is to characterize those network configurations that give rise to optimal field estimates. This is the problem that we consider in this paper.

Literature review: Kriging [1], [2] is a standard geostatistical technique used to produce estimates of spatial processes based on data collected at a finite number of locations. An advantage of kriging over other spatial interpolation methods is that it provides a measure of the uncertainty associated to the estimator. The optimal design literature [3], [4] deals with the design of experiments to optimize the resulting statistical estimation. Of particular interest are the notions of G-optimality, minimizing the maximum prediction error, and D-optimality, minimizing the generalized prediction variance.

The work [5] introduces performance metrics for optimal estimation in oceanographic research. The works [6], [7] propose distributed optimal estimation strategies for deterministic fields, when the measurements taken by individual agents are uncorrelated. In [8], the emphasis is on finding optimal agent trajectories along a given interval of time among a parameterized set of trajectories. Here, instead, we focus on optimal network configurations for the estimation of the random field at a single snapshot. In our technical approach, we have been inspired by [9], which considers the problem of minimizing the maximum uncertainty over a discrete space and shows that minimax configurations are asymptotically optimal as the correlation between any two distinct points vanishes. Minimax configurations minimize the maximum distance to the nearest agent from any point in space. Here, we make the connection to Voronoi partitions of continuous spaces, a classical notion in computational geometry [10]. In [11], circumcenter and incenter Voronoi configurations are defined, and distributed motion coordination algorithms are introduced to asymptotically steer the network towards them.

Statement of contributions: In this paper, we consider two performance metrics for optimal placement of mobile sensor networks based on kriging. We first characterize the continuity properties of the mean-squared error of the simple kriging estimator as a function of the network configuration. In the case of zero measurement error, previous results in the optimal design literature have avoided this problem by optimizing over a discrete set of possible configurations, while we consider the continuous space of all agent locations within the region. Next, we define our first optimality criterion, the maximum prediction error of the kriging predictor, and study its critical points asymptotically, as the correlation between any two distinct points vanishes. We define a second optimality criterion as a form of D-optimality, the generalized variance of the kriging predictor within a bounded region, and study the critical points within the same asymptotic framework. Our main results are showing that, for the simple kriging predictor, circumcenter Voronoi configurations are asymptotically optimal for the maximum prediction error over the environment, while incenter Voronoi configurations are asymptotically optimal for the generalized variance. In general, it is difficult to obtain exactly the configurations that optimize these objective functions. Our results are relevant to the extent that they guarantee that, for scenarios with small correlation between distinct points, circumcenter and incenter Voronoi configurations are optimal for appropriate measures of uncertainty. The network can achieve these desirable configurations by executing simple distributed dynamical systems, as illustrated in simulations.

Organization: The paper is organized as follows. Section II introduces basic notions from computational geometry and presents a brief overview of the kriging estimation procedure. Section III states the problem of interest. We

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present our main results in Section IV on the optimality of circumcenter and incenter Voronoi configurations. Section V presents simulations to illustrate our results. Finally, Section VI gathers our conclusions and ideas for future work. For reasons of space, some of the proofs have been omitted. The interested reader may find them online in [12].

II. PRELIMINARIES

Let us start with some notation. Let $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the set of reals, positive reals, and nonnegative reals, respectively. We are concerned with operations on a compact and connected set $\mathcal{D}$ of Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$. For $p, q \in \mathbb{R}^d$, we let $]p, q[= \{\lambda p + (1-\lambda) q \mid \lambda \in]0,1[\}$ denote the *open segment* with extreme points p and q. For $p \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$, $\overline{B}(p, r)$ denotes the *closed ball* of radius r centered at p. We denote by $|S|$ and ∂S the cardinality and the boundary of a set S, respectively. A *convex polytope* is the convex hull of a finite point set. For $S \subset \mathbb{R}^d$ bounded, the *circumcenter* $\mathrm{CC}(S)$ and *circumradius* $\mathrm{CR}(S)$ are, respectively, the center and radius of the smallest-radius ball enclosing S. The incenter set $\mathrm{IC}(S)$ of S is the set of the centers of maximum-radius d-spheres contained in S. The inradius $\mathrm{IR}(S)$ of S is the common radius of these balls.

We consider tuples or ordered sets of possibly coincident points, $P = (p_1, \dots, p_n) \in (\mathbb{R}^d)^n$. We refer to such an element as a *configuration*. Let $\mathfrak{P}(S)$ (respectively $\mathbb{F}(S)$) denote the collection of subsets (respectively, finite subsets) of S. We denote an element of $\mathbb{F}(\mathbb{R}^d)$ by $\mathcal{P} = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$, where $p_1, \dots, p_n$ are distinct points in $\mathbb{R}^d$. Let $i_{\mathbb{F}} : (\mathbb{R}^d)^n \to \mathbb{F}(\mathbb{R}^d)$ be the natural immersion, i.e., $i_{\mathbb{F}}(P)$ contains only the distinct points in $P = (p_1, \dots, p_n)$. Note that the cardinality of $i_{\mathbb{F}}(p_1, \dots, p_n)$ is in general less than or equal to n. Let S_{coinc} be the set of all tuples in $(\mathbb{R}^d)^n$ which contain at least one coincident pair of points, that is,

$$S_{\text{coinc}} = \{(p_1, \dots, p_n) \in (\mathbb{R}^d)^n \mid p_i = p_j \text{ for some } i, j \in \{1, \dots, n\}, i \neq j\}.$$

Let $\|\cdot\|$ denote the Euclidean distance function on $\mathbb{R}^d$. Define the distance $\mathrm{d} : \mathbb{R}^d \times \mathfrak{P}(\mathcal{D}) \to \mathbb{R}$ from a point in $\mathbb{R}^d$ to a set of points in $\mathcal{D}$ by $\mathrm{d}(s, \mathcal{P}) = \inf_{p \in \mathcal{P}} \{\|s - p\|\}$, and let $\mathrm{mds} : \mathbb{R}^d \times \mathfrak{P}(\mathcal{D}) \to \mathfrak{P}(\mathcal{D})$ be the *minimum distance set* map defined by $\mathrm{mds}(s, \mathcal{P}) = \{p \in \mathcal{P} \mid \|s - p\| = \mathrm{d}(s, P)\}$.

A. Voronoi partitions and multicenter problems

Here we present some relevant concepts on Voronoi diagrams and refer the reader to [10], [13] for comprehensive treatments. A *partition* of $\mathcal{D}$ is a collection of n polygons $\mathcal{W} = \{W_1, \dots, W_n\}$ with disjoint interiors whose union is $\mathcal{D}$. The *Voronoi partition* $\mathcal{V}(P) = (V_1(P), \dots, V_n(P))$ of $\mathcal{D}$ generated by the points $P = (p_1, \dots, p_n)$ is defined by

$$V_i(P) = \{q \in \mathcal{D} \mid \|q - p_i\| \leq \|q - p_j\|, \ \forall j \neq i\}.$$

We say that P is a *circumcenter Voronoi configuration* if $p_i = \mathrm{CC}(V_i(P))$, for all $i \in \{1, \dots, n\}$, and that P is an *incenter Voronoi configuration* if $p_i \in \mathrm{IC}(V_i(P))$, for all $i \in \{1, \dots, n\}$. An incenter Voronoi configuration is *isolated* if there exists a neighborhood around it in $\mathcal{D}^n$ which does not contain any other incenter Voronoi configuration.

Consider the *disk-covering* and *sphere-packing multicenter* functions defined by

$$\mathcal{H}_{\text{DC}}(P) = \max_{s \in \mathcal{D}} \{\mathrm{d}(s, i_{\mathbb{F}}(P))\},$$
$$\mathcal{H}_{\text{SP}}(P) = \min_{i \neq j \in \{1, \dots, n\}} \left\{\frac{1}{2}\|p_i - p_j\|, \mathrm{d}(p_i, \partial\mathcal{D})\right\}.$$

We are interested in the configurations that optimize these multicenter functions. The minimization of $\mathcal{H}_{\text{DC}}$ corresponds to minimizing the largest possible distance of any point in $\mathcal{D}$ to one of the agents' locations given by $p_1, \dots, p_n$. We refer to it as the as the *multi-circumcenter problem*. The maximization of $\mathcal{H}_{\text{SP}}$ corresponds to the situation where we are interested in maximizing the coverage of the area $\mathcal{D}$ in such a way that the radius of the generators do not overlap (in order not to interfere with each other) or leave the environment. We refer to it as the *multi-incenter problem*. It is useful to define the *index function* $N : \mathcal{D}^n \to \mathbb{N}$ as

$$N(P) = \left| \underset{p_i \neq p_j}{\operatorname{argmin}} \left\{\frac{1}{2}\|p_i - p_j\|, \mathrm{d}(p_i, \partial\mathcal{D})\right\} \right|.$$

B. Spatial prediction via simple kriging

This section reviews the geostatistical kriging procedure for the estimation of spatial processes, see e.g., [1], [14]. A random process Z is *second-order stationary* if it has constant mean, $E(Z(s)) = \mu$, and its covariance is of the form $\mathrm{Cov}(Z(p_1), Z(p_2)) = C(p_1, p_2)$, where $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ is a valid covariance function that only depends on the difference $p_1 - p_2$. We focus on *isotropic* covariance functions, which satisfy $C(p_1, p_2) = g(\|p_1 - p_2\|)$, for some decreasing function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. The covariance matrix of the set of points $p_1, \dots, p_n \in \mathcal{D}$ is $\boldsymbol{\Sigma} = \Sigma(P) = [C(p_i, p_j)]_{i,j=1}^n \in \mathbb{R}^{n \times n}$. When it is clear from the context, we use bold face to denote explicit dependence on P. We define $c : \mathcal{D} \times \mathcal{D}^n \to \mathbb{R}^n$ to be the vector of covariances between a point $s \in \mathcal{D}$ and the locations in P, i.e., $\boldsymbol{c} = c(s, P) = (C(s, p_1), \dots, C(s, p_n))^T$. The associated correlation function, $\rho : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ is defined as

$$\rho(p_1, p_2) = \frac{C(p_1, p_2)}{\sqrt{C(p_1, p_1)}\sqrt{C(p_2, p_2)}} = \frac{g(\|p_1 - p_2\|)}{g(0)}.$$

Throughout the paper, we make the following assumptions on the model for the spatial random process Z of interest. We assume that Z is of the form

$$Z(s) = \mu(s) + \delta(s), \quad s \in \mathcal{D}, \tag{1}$$

and that the mean function μ is known. Also, δ is a zero-mean second-order stationary random process with a known decreasing isotropic covariance function, g. We further assume that g is everywhere differentiable. Some examples of such functions are the exponential, cubic, spherical, modified Bessel, and rational quadratic covariance functions.

Assume measurement data $\boldsymbol{y} = (Y(p_1), \dots, Y(p_n))^T$ are corrupted with error such that

$$Y(p_i) = Z(p_i) + \epsilon_i, \qquad \epsilon_i \overset{\text{iid}}{\sim} N(0, \tau^2), \tag{2}$$

where $\tau^2 \geq 0$. The assumption that the errors ϵ_i, $i \in \{1, \dots, n\}$ are independent and identically distributed corresponds to the fact that the robotic network is equipped with identical sensors. The no-error scenario is the one most widely studied in the geostatistics literature. In the error case, the covariance between $Y(p_i)$ and $Y(p_j)$ is given by

$$\operatorname{Cov}(Y(p_i), Y(p_j)) = \begin{cases} C(p_i, p_j) + \tau^2, & \text{if } i = j, \\ C(p_i, p_j), & \text{if } i \neq j. \end{cases}$$

Note that the covariance matrix of P with respect to the noisy process Y may be written $\mathbf{\Sigma}_\tau = \Sigma_\tau(P) = \mathbf{\Sigma} + \tau^2 I_n$, where I_n denotes the $n \times n$ identity matrix. The no-error case is recovered by setting $\tau^2 = 0$.

The *simple kriging predictor* at $s \in \mathcal{D}$ is the predictor that minimizes the mean-squared prediction error $\sigma^2(s; p_1, \dots, p_n) = E(Z(s) - p(s; Y(p_1), \dots, Y(p_n)))^2$ among all unbiased predictors of the form $p(s; Y(p_1), \dots, Y(p_n)) = \sum_{i=1}^n l_i Y(p_i) + k$. The explicit expression of the simple kriging predictor at $s \in \mathcal{D}$ is

$$\hat{p}_{\mathrm{SK}}(s; Y(p_1), \dots, Y(p_n)) = \mu(s) + \boldsymbol{c}^T \mathbf{\Sigma}_\tau^{-1} (\boldsymbol{y} - \boldsymbol{\mu}), \quad (3)$$

where $\boldsymbol{\mu} = (\mu(p_1), \dots, \mu(p_n))^T$. The mean-squared prediction error of $\hat{p}_{\mathrm{SK}}$ at $s \in \mathcal{D}$ is

$$\sigma^2(s; p_1, \dots, p_n) = g(0) - \boldsymbol{c}^T \mathbf{\Sigma}_\tau^{-1} \boldsymbol{c}. \quad (4)$$

Note that σ^2 is invariant under permutations of $p_1, \dots, p_n$.

III. Problem statement

Consider a network of n agents evolving in a convex polytope $\mathcal{D} \subset \mathbb{R}^d$ according to the first-order dynamics $\dot{p}_i = u_i$, $i \in \{1, \dots, n\}$. Assume each agent is equipped with a point-sized footprint sensor, and can take a noisy measurement $Y(p_i)$ as in (2) of the spatial process Z at its current position p_i. A natural objective is to select locations to take measurements in such a way as to minimize the uncertainty in the estimate of the spatial process. Here, we consider objectives inspired by the notions of G- and D-optimality from optimal design [1], [3].

The *maximum prediction error* is

$$\begin{aligned} \mathcal{M}(p_1, \dots, p_n) &= \max_{s \in \mathcal{D}} \sigma^2(s; p_1, \dots, p_n) \\ &= g(0) - \min_{s \in \mathcal{D}} \{\boldsymbol{c}^T \mathbf{\Sigma}_\tau^{-1} \boldsymbol{c}\}. \end{aligned} \quad (5a)$$

Note that $\mathcal{M}$ corresponds to a "worst-case scenario," where we consider locations with the maximum kriging mean-squared error. Let us make an important observation about the well-posedness of $\mathcal{M}$. Under noisy measurements, i.e., $\tau^2 > 0$, the function σ^2 is well-defined for any $s \in \mathcal{D}$ and $(p_1, \dots, p_n) \in \mathcal{D}^n$. Indeed, the dependence of σ^2 on the network configuration is continuous, and hence, $\mathcal{M}$ is also well-defined. However, when no measurement noise is present, i.e., $\tau^2 = 0$, then the matrix $\mathbf{\Sigma}_\tau = \mathbf{\Sigma}$ in (4) is not invertible for configurations that belong to S_{coinc}, and therefore, it is not clear what the value of σ^2 is. Proposition IV.2 below states that, in the no measurement noise case, σ^2 is a continuous function of the network configuration under suitable technical conditions on the spatial field covariance.

Before presenting the second objective function, we need to introduce some notions. Note that the variance of the simple kriging predictor is given by the expression $\boldsymbol{c}^T \mathbf{\Sigma}_\tau^{-1} \boldsymbol{c}$, while the generalized variance [15] is considered to be the determinant of the covariance matrix $\mathbf{\Sigma}_\tau^{-1}$. Minimizing the determinant of $\mathbf{\Sigma}_\tau^{-1}$ is equivalent to minimizing $-|\mathbf{\Sigma}_\tau|$, where $|\cdot|$ denotes the determinant. For discrete state spaces, it can be shown [9] that configurations which maximize the minimum distance between agents asymptotically minimize $-|\mathbf{\Sigma}_\tau|$ in the limit of near independence, but this tends to place agents on the boundary of $\mathcal{D}$. Since we are only interested in predictions over $\mathcal{D}$, we would like a notion of optimality which penalizes agents too close to the boundary as it does agents too close to each other. This can be achieved as follows. Let $\gamma : \mathcal{D} \to \mathbb{R}^d$ map a point in $\mathcal{D}$ to its mirror image reflected across the nearest boundary of $\mathcal{D}$. Formally,

$$\gamma(s) \in s + 2 \left(\operatorname*{argmin}_{s^* \in \partial \mathcal{D}} \{\|s^* - s\|\} - s \right).$$

Note that $\gamma(s)$ is in general not unique, and is not a smooth function of s. However, $\|s - \gamma(s)\|$ is smooth, and is the same for all values of $\gamma(s)$. Now consider minimizing the determinant of the simple kriging predictor which would result if we had data from all agents as well as their reflections. The *generalized prediction variance* is then

$$\mathcal{B}(p_1, \dots, p_n) = -\left| \Sigma_\tau(p_1, \dots, p_n, \gamma(p_1), \dots, \gamma(p_n)) \right|. \quad (5b)$$

Note that since $\mathcal{B}$ does not require inversion of the covariance matrix, it is always well-posed. Our main goal is to find network configurations that optimize $\mathcal{M}$ and $\mathcal{B}$.

IV. Optimal configurations for spatial prediction

In this section, we provide several results that characterize the optimal network configurations for the objective functions $\mathcal{M}$ and $\mathcal{B}$. In Section IV-A, we show that minima of $\mathcal{M}$ cannot be in S_{coinc}. This is used in Section IV-B where we show that circumcenter and incenter Voronoi configurations are asymptotically optimal for $\mathcal{M}$ and $\mathcal{B}$, respectively.

A. Coincident configurations are not minima of $\mathcal{M}$

We examine the effect of the location of a subset of agents on the mean-squared error. We are interested in comparing $\sigma^2(s; P)$ against $\sigma^2(s; i_{\mathbb{F}}(P))$ for $P \in S_{\mathrm{coinc}}$. The following result provides a useful decomposition of σ^2.

Lemma IV.1 *The simple kriging mean-squared error function may be written in the form*

$$\sigma^2(s; P) = \sigma^2(s; \overline{P}) - \frac{\left(\mathcal{N}(s, p_1; \overline{P})\right)^2}{\sigma^2(p_1; \overline{P}) + \tau^2}, \quad (6)$$

with $\mathcal{N}(s, p_1; \overline{P}) = C(s, p_1) - c^T(s, \overline{P}) \Sigma_\tau(\overline{P})^{-1} c(p_1, \overline{P})$ *and* $\overline{P} = (p_2, \dots, p_n) \in \mathcal{D}^{n-1}$.

This fact may be proven using [16, Proposition 8.2.4] for the inverse of a partitioned symmetric matrix. Equation (6) may be applied repeatedly to isolate the effects of any subset of locations in P. The following proposition is useful for considering the behavior of $\mathcal{M}$ as agents move around $\mathcal{D}$.

Proposition IV.2 (Continuity of simple kriging predictor error) *Let* $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *be an isotropic covariance function,* $C(p_1, p_2) = g(\|p_1 - p_2\|)$, *with* $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

differentiable. Assume $g'(0) \neq 0$ and $\tau^2 = 0$. Then, for $s \in \mathcal{D}$, the mean-squared simple kriging prediction error $(p_1, \dots, p_n) \mapsto \sigma^2(s; p_1, \dots, p_n)$ is continuous. Moreover, $\sigma^2(s; P) = \sigma^2(s; i_{\mathbb{F}}(P))$ for $P \in S_{coinc}$.

Under the assumptions of Proposition IV.2, we can extend the mean squared error function by continuity to include configurations in S_{coinc}. With a slight abuse of notation, in the case of no measurement error, we use $\sigma^2(s; P)$ to denote $\sigma^2(s; i_{\mathbb{F}}(P))$ for $P \in S_{\text{coinc}}$.

Proposition IV.3 (Minima of $\mathcal{M}$ are not in S_{coinc}) *Let $P^\dagger \in \mathcal{D}^n$ be a strict local minimum of the map $P \mapsto \mathcal{M}(P)$. Under the assumptions of Proposition IV.2, $P^\dagger \notin S_{coinc}$.*

Proof: We proceed by contradiction. Assume $P^\dagger \in S_{\text{coinc}}$. Consider a configuration $P \in \mathcal{D}^n \setminus S_{\text{coinc}}$ in a neighborhood of $P^\dagger$ such that $i_{\mathbb{F}}(P^\dagger) \subset i_{\mathbb{F}}(P)$. Let $s, s^\dagger \in \mathcal{D}$ such that $\mathcal{M}(P) = \sigma^2(s; P)$ and $\mathcal{M}(P^\dagger) = \sigma^2(s^\dagger; P^\dagger)$. Using Lemma IV.1 and Proposition IV.2, one can deduce that $\sigma^2(s; P^\dagger) \geq \sigma^2(s; P)$. By the definition of $\mathcal{M}$, $\sigma^2(s^\dagger; P^\dagger) \geq \sigma^2(s; P^\dagger)$. Therefore $\mathcal{M}(P^\dagger) = \sigma^2(s^\dagger; P^\dagger) \geq \sigma^2(s; P^\dagger) \geq \sigma^2(s; P) = \mathcal{M}(P)$, which is a contradiction. ■

B. Asymptotic optimality of multicenter configurations

Let us consider the objective functions $\mathcal{M}$ and $\mathcal{B}$ introduced in Section III but with covariance function C^k, $k \in \mathbb{N}$. This is equivalent to considering the correlation, ρ^k. As k grows, the correlation between distinct points in $\mathcal{D}$ vanishes. To ease the exposition, we denote by $\boldsymbol{c}^{(k)}$, respectively $\boldsymbol{\Sigma}_\tau^{(k)}$, the vector $\boldsymbol{c}$, respectively the matrix $\boldsymbol{\Sigma}_\tau$, with each element raised to the kth power. Similarly, let $\mathcal{M}^{(k)}, \mathcal{B}^{(k)} : D^n \to \mathbb{R}$,

$$\mathcal{M}^{(k)}(p_1, \dots, p_n) = g^k(0) - \min_{s \in \mathcal{D}} \{(\boldsymbol{c}^{(k)})^T (\boldsymbol{\Sigma}_\tau^{(k)})^{-1} \boldsymbol{c}^{(k)}\},$$

$$\mathcal{B}^{(k)}(p_1, \dots, p_n) = -\left| \Sigma_\tau^{(k)}\left(p_1, \dots, p_n, \gamma(p_1), \dots, \gamma(p_n)\right) \right|.$$

First we establish a result on the cardinality of the minimum distance set. Let $C_{\text{mds}} : \mathbb{R}^d \times \mathcal{D}^n \to \mathbb{R}$ be defined by

$$C_{\text{mds}}(s, P) = C(s, p) \quad \text{for any } p \in \text{mds}(s, P).$$

Note that C_{mds} is well-defined.

Proposition IV.4 (Cardinality of minimum distance set) *Let the covariance function C be continuous. For $P \in \mathcal{D}^n \setminus S_{coinc}$, one has*

$$\min_{s \in \mathcal{D}} \{C_{\text{mds}}(s, P) \, |\text{mds}(s, P)|\} = \min_{s \in \mathcal{D}} \{C_{\text{mds}}(s, P)\}.$$

We are now ready to prove one of the main results of the paper. The proof follows a similar line of reasoning to [9].

Theorem IV.5 (Minima of $\mathcal{M}$ under near independence) *Let $P_{mcc} \in \mathcal{D}^n$ be a global minimizer of the multi-circumcenter problem. Then, as $k \to \infty$, P_{mcc} asymptotically globally optimizes $\mathcal{M}^{(k)}$, i.e., $\mathcal{M}^{(k)}(P_{\text{mcc}})$ approaches a global minimum.*

Proof: Note that minimizing $\mathcal{M}^{(k)}$ is equivalent to finding the tuples P which maximize the function $L^{(k)} : \mathcal{D}^n \to \mathbb{R}$ defined as

$$L^{(k)}(P) = \min_{s \in \mathcal{D}} \left\{ (c^{(k)}(s, P))^T (\Sigma_\tau^{(k)}(P))^{-1} (c^{(k)}(s, P)) \right\}.$$

Let $\lambda_{\min}$ and $\lambda_{\max} : \mathcal{D}^n \times \mathbb{R} \to \mathbb{R}$ be such that $\lambda_{\min}(P, k)$, $\lambda_{\max}(P, k)$ denote, respectively, the minimum and the maximum eigenvalue of $\Sigma_\tau^{(k)}(P)$. We can see that $L^{(k)}(P)$ is bounded above by $\lambda_{\max}(P, k) \sum_{p \in P} C(s, p)^{2k}$ and below by $\lambda_{\min}(P, k) \sum_{p \in P} C(s, p)^{2k}$. For a given s, in terms of the minimum distance set we can write

$$\begin{aligned} \sum_{p \in P} C(s, p)^{2k} &= \sum_{p \in \text{mds}(s, P)} C(s, p)^{2k} + \sum_{p \in P \setminus \text{mds}(s, P)} C(s, p)^{2k} \\ &= |\text{mds}(s, P)| \, C_{\text{mds}}(s, P)^{2k} + \sum_{p \in P \setminus \text{mds}(s, P)} C(s, p)^{2k}. \end{aligned}$$

As $k \to \infty$ the elements in the minimum distance set dominate, so we are left with

$$\begin{aligned} \sum_{p \in P} C(s, p)^{2k} = |\text{mds}(s, P)| \, C_{\text{mds}}(s, P)^{2k} + \\ + o(C_{\text{mds}}(s, P)^{2k}). \end{aligned}$$

Note from Proposition IV.4 that

$$\min_{s \in \mathcal{D}} \{|\text{mds}(s, P)| \, C_{\text{mds}}(s, P)\} = \min_{s \in \mathcal{D}} \{C_{\text{mds}}(s, P)\},$$

so we can write

$$\min_{s \in \mathcal{D}} \{ \sum_{p \in P} C(s, p)^{2k} \} = \min_{s \in \mathcal{D}} \left\{ C_{\text{mds}}(s, P)^{2k} \, (1 + o(1)) \right\}.$$

Consider, then, comparing an arbitrary configuration P^* against a global minimizer of $\mathcal{H}_{\text{DC}}$, say P_{mcc}. In the zero measurement error case, by Proposition IV.3, we can assume without loss of generality that $P^* \notin S_{\text{coinc}}$. Therefore, no matter what the value of τ is, we can safely use the eigenvalues of $(\boldsymbol{\Sigma}_\tau^{(k)})^{-1}$ to provide bounds. Specifically,

$$\begin{aligned} \frac{L^{(k)}(P^*)}{L^{(k)}(P_{\text{mcc}})} \leq \qquad (7) \\ \frac{\lambda_{\max}(P^*, k) \min_{s \in \mathcal{D}} \left\{ C_{\text{mds}}(s, P^*)^{2k} \, (1 + o(1)) \right\}}{\lambda_{\min}(P_{\text{mcc}}, k) \min_{s \in \mathcal{D}} \left\{ C_{\text{mds}}(s, P_{\text{mcc}})^{2k} \, (1 + o(1)) \right\}}. \end{aligned}$$

Next we take a closer look at the eigenvalues. Note that if we divide $\Sigma_\tau^{(k)}(P)$ by the common factor of $\alpha^k = (g(0) + \tau^2)^k \in \mathbb{R}$, the resulting correlation matrix becomes diagonal in the limit. This gives us $\lim_{k \to \infty} \Sigma_\tau^{(k)}(P)/\alpha^k = I_n$, and one can see that $\lambda_{\min}(P, k)/\alpha^k$ and $\lambda_{\max}(P, k)/\alpha^k$ both tend to 1 for any configuration P. Finally, since P_{mcc} minimizes the maximum distance to any point $s \in \mathcal{D}$, it maximizes the minimum covariance, so for any $P \in \mathcal{D}^n$, $\min_{s \in \mathcal{D}} C_{\text{mds}}(s, P) \leq \min_{s \in \mathcal{D}} C_{\text{mds}}(s, P_{\text{mcc}})$. Thus the ratio (7) is bounded by $1 + o(1)$. Therefore, in the limit as $k \to \infty$, minimizing $\mathcal{M}^{(k)}$ is equivalent to solving the multi-circumcenter problem. ■

The proof of this theorem can be reproduced for local minimizers of the multi-circumcenter problem.

Corollary IV.6 *Let $P_{\text{mcc}} \in \mathcal{D}^n$ be a local minimizer of the multi-circumcenter problem. Then, as $k \to \infty$, P_{mcc} asymptotically optimizes $\mathcal{M}^{(k)}$, i.e., $\mathcal{M}^{(k)}(P_{\text{mcc}})$ approaches a minimum.*

According to [11], under certain technical conditions, solutions to the multi-circumcenter problem are circumcenter Voronoi configurations. Next, let us present a similar asymptotic result for the generalized prediction variance.

Theorem IV.7 (Minima of $\mathcal{B}$ under near independence) *Let $P_{\rm mic} \in \mathcal{D}^n$ be a global maximizer of the multi-incenter problem with lowest index. Then, as $k \to \infty$, $P_{\rm mic}$ asymptotically globally optimizes $\mathcal{B}^{(k)}$, i.e., $\mathcal{B}^{(k)}(P_{\rm mic})$ approaches a global minimum.*

Proof: Expanding the objective function for asymptotically dominant terms, we may write

$$\mathcal{B}^{(k)}(P) = -(g(0)+\tau^2)^{k(2n)} + \left(g(0)+\tau^2\right)^{k(2n-2)} J^{(k)}(P) + + o\left(\left(g(0)+\tau^2\right)^{k(2n-2)} J^{(k)}(P)\right),$$

where

$$J^{(k)}(P) = \sum_{i\neq j} g(\|p_i - p_j\|)^{2k} + \sum_{i,j=1}^{n} g(\|p_i - \gamma(p_j)\|)^{2k} + + \sum_{i\neq j} g(\|\gamma(p_i) - \gamma(p_j)\|)^{2k}.$$

Asymptotically all but the largest terms in $J^{(k)}(P)$ will drop out, and minimizing $\mathcal{B}^{(k)}(P)$ becomes equivalent to minimizing those terms. The largest terms in $J^{(k)}(P)$ correspond to the shortest distance between the locations of either the agents or their reflected images. For any two agent locations, $p_i, p_j \in \mathcal{D}$, and any of their reflections $\gamma(p_i), \gamma(p_j)$ the minimum distance between any two of the four points can be reduced to $\min\{\|p_i - p_j\|, \|p_i - \gamma(p_i)\|, \|p_j - \gamma(p_j)\|\}$ (note that this is not in general true for non-convex domains). Thus the shortest distance between agents in P and their reflections may be expressed as $2\mathcal{H}_{\rm SP}(P)$, though the index of P might be larger than 1. Therefore we have $J^{(k)}(P) = N(P)\left(g(2\mathcal{H}_{\rm SP}(P))^{2k}\right)(1+o(1))$. Consider comparing an arbitrary configuration, $P^* \in \mathcal{D}^n$ against $P_{\rm mic}$. We have

$$\frac{J^{(k)}(P_{\rm mic})}{J^{(k)}(P^*)} = \frac{N(P_{\rm mic})\left(g(2\mathcal{H}_{\rm SP}(P_{\rm mic}))^{2k}\right)(1+o(1))}{N(P^*)\left(g(2\mathcal{H}_{\rm SP}(P^*))^{2k}\right)(1+o(1))}.$$

If P^* is not a global solution of the multi-incenter problem, $\mathcal{H}_{\rm SP}(P_{\rm mic}) > \mathcal{H}_{\rm SP}(P^*)$, and since g is decreasing, then

$$\lim_{k\to\infty} \frac{J^{(k)}(P_{\rm mic})}{J^{(k)}(P^*)} = 0.$$

If P^* is a global solution of the multi-incenter problem, then, using the fact that $P_{\rm mic}$ has the lowest index among all of them, we deduce $\frac{J^{(k)}(P_{\rm mic})}{J^{(k)}(P^*)} \leq 1 + o(1)$. ■

Note that the proof of the theorem can be reproduced for isolated local maximizers of the multi-incenter problem.

Corollary IV.8 *Let $P_{\rm mic} \in \mathcal{D}^n$ be an isolated local maximizer of the multi-incenter problem. Then, as $k \to \infty$, $P_{\rm mic}$ asymptotically optimizes $\mathcal{B}^{(k)}$, i.e., $\mathcal{B}^{(k)}(P_{\rm mic})$ approaches a minimum.*

According to [11], under certain technical conditions, solutions to the multi-incenter problem are incenter Voronoi configurations.

C. Distributed coordination algorithms

In this section, we present coordination algorithms that steer the network towards circumcenter and incenter Voronoi configurations. We do this following the exposition in [11]. In light of the results in Section IV-B, this enables the network to perform a spatial prediction which is asymptotically optimal as $k \to \infty$.

Let us assume each agent can move according to a first-order dynamical model $\dot{p}_i = u_i$, $i \in \{1, \dots, n\}$. Consider the following coordination algorithms

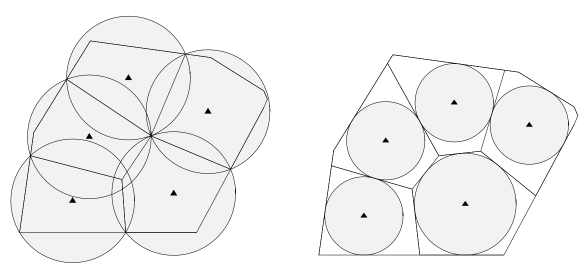
$$\dot{p}_i = \mathrm{CC}(V_i(P)) - p_i, \tag{8a}$$

$$\dot{p}_i \in \mathrm{IC}(V_i(P)) - p_i, \tag{8b}$$

for each $i \in \{1, \dots, n\}$. Note that (8b) is a differential inclusion. We understand its solutions in the Filippov sense [17]. Both coordination algorithms are Voronoi distributed, meaning that each agent only requires information from its Voronoi neighbors in order to execute its control law. The equilibrium points of the flow (8a) are the circumcenter Voronoi configurations, whereas the equilibrium points of the flow (8b) are incenter Voronoi configurations. Furthermore, the evolution of $\mathcal{H}_{\rm DC}$ along (8a) is monotonically decreasing, while the evolution of $\mathcal{H}_{\rm SP}$ along (8b) is monotonically increasing. The convergence properties of these coordination algorithms, as well as alternative flows with similar distributed properties that can also be used to steer the network to center Voronoi configurations, are studied in [11].

V. Simulations

We performed simulations for $\mathcal{M}$ and $\mathcal{B}$ with $n = 5$ agents. We used as domain $\mathcal{D}$ the convex polygon with vertices $\{(0,.1), (2.5,.1), (3.45,1.6), (3.5,1.7), (3.45,1.8), (2.7,2.2), (1,2.4), (0.2,1.3)\}$ and as isotropic covariance the one defined via $C : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $C(s_1, s_2) = e^{-\frac{1}{5}\|s_1 - s_2\|}$. Note that in simple kriging the mean function does not play a role in determining the optimal network configurations. Figure 1 shows the multicenter configurations obtained with (8).



(a) (b)

Fig. 1. Multicenter configurations found (a) using the flow (8a) and (b) using the flow (8b).

A. Analysis of simulations for $\mathcal{M}^{(k)}$

Using $\mathcal{M}^{(1)}$ we ran over 1000 random trials, each time running a gradient descent algorithm, and chose the local minimum configuration with the smallest value of $\mathcal{M}^{(1)}$ to be our approximation of a global minimum. Following (8a) from this configuration P_*, we generated the multi-circumcenter configuration depicted in Figure 1(a). For increasing values of k, we ran a gradient descent of $\mathcal{M}^{(k)}$ to find the best local configuration near P_*. For comparison, we also plotted the performance of a random configuration which was not a local minimum. Figure 2 illustrates the result in Theorem IV.5. At around $k = 15$, the performance of the circumcenter

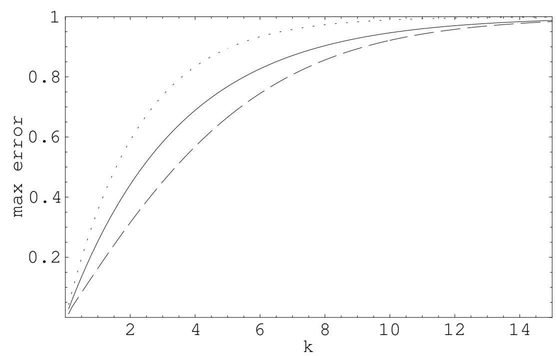


Fig. 2. Value of $\mathcal{M}^{(k)}$ for multi-circumcenter (solid), approximated global minimum (dashed), and random (dotted) configurations of 5 agents for increasing values of k. The covariance function is exponential.

Voronoi configuration becomes almost identical to the one of the minimizer of $\mathcal{M}^{(k)}$.

B. Analysis of simulations for $\mathcal{B}^{(k)}$

Using $\mathcal{B}^{(1)}$ we ran over 1000 random trials, each time running a gradient descent algorithm, and chose the local minimum configuration with the smallest value of $\mathcal{B}^{(1)}$ to be our approximation of a global minimum. Following (8b) from this configuration P_*, we generated the multi-incenter configuration depicted in Figure 1(b). For increasing values of k, we ran a gradient descent of $\mathcal{B}^{(k)}$ to find the best local configuration near P_*. For comparison, we also plotted the performance of a random configuration which was not a local minimum. Figure 3 illustrates the result stated in Theorem IV.7. The performance of the minimizer of $\mathcal{B}^{(k)}$ and

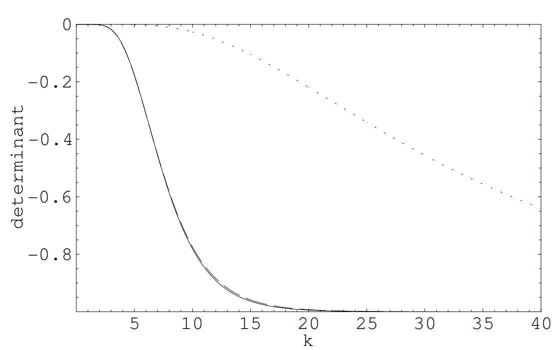


Fig. 3. Value of $\mathcal{B}^{(k)}$ for multi-incenter (solid), approximated global minimum (dashed), and random (dotted) configurations of 5 agents for increasing values of k. The covariance function is exponential. Although the performance of the global and multi-incenter configurations look identical, the configurations are different at each k.

of the incenter Voronoi configuration are almost identical, even though at each k the configurations are different.

VI. Conclusions

We have used simple kriging as a metric for optimal placement of mobile sensor networks estimating random fields. We have shown that under the assumption of near independence, circumcenter configurations minimize the maximum prediction error and incenter configurations minimize the generalized prediction variance. Under limited time or energy resources, or as a starting point for further exploration, a group of robotic sensors might begin by moving toward these configurations to start the estimation procedure.

Future work will explore: (i) regarding the asymptotic analysis, the determination of bounds on the parameter k that guarantee that multicenter Voronoi configurations achieve a desired level of performance, (ii) the extension of the results to similar error metrics for the universal kriging predictor, and (iii) the characterization of the agent trajectories (rather than configurations) that provide optimal estimates of the random field. Consideration will also be given to dynamic fields, and to distributed methods for data fusion.

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