Use of numerical simulation to solve the Couette flow problem

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Abstract
The Couette flow is a simplified 2d fluid dynamics problem. In this project I describe a numerical simulation I developed in order to look at turbulence in this flow.

1 Introduction

The Couette flow is a 2-dimensional fluid flow with toroidal boundaries along one axis, and fixed boundaries along the other. See Figure 1 and Figure 2.

$u$ and $v$ are the velocity components of the flow, while $x$ and $y$ are the coordinates. Along the $y$ boundaries the fluid is allowed to slip, but not interpenetrate.

Rotating body

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{couette_flow.png}
\caption{3d setup of Couette flow}
\end{figure}

We will let $\vec{u} = (u, v)$.
The general equations governing this system are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial \rho}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + F(y)$$

and

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial \rho}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

with the constraint that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$
Figure 2: 2d setup of Couette flow and axis labels

2 Laminar flow

Under the assumption that flow is independent of $x$, this becomes

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} = -\frac{\partial \rho}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + F(y)$$

and

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} = -\frac{\partial \rho}{\partial x} + \nu \frac{\partial^2 v}{\partial y^2}$$

with the constraint that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

with the constraint that

$$\frac{\partial v}{\partial y} = 0$$

Combining these gives

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} = -\frac{\partial \rho}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + F(y)$$

and $v = 0$ and $\frac{\partial \rho}{\partial x} = 0$ which implies

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + F(y).$$

The resulting equation is linear, and can be solved analytically quite easily. If we let $F(y) = \sum_{k=0}^{\infty} a_k e^{2\pi k y}$ and $u(t) = \sum_{k=0}^{\infty} u_k(t) e^{2\pi k y}$ we have $\frac{d}{dt} u_k = -\nu 4 k^2 \pi^2 u_k + a_k$, which gives the dynamics of $u_k(t) = \frac{a_k}{\nu 4 k^2 \pi^2} \left( (u_k(0) - \frac{a_k}{\nu 4 k^2 \pi^2}) e^{-rt} \right)$ where $r = \nu (4 k^2 \pi^2)$.

There are three things of note here (assuming the forcing is only in $x$, which is true for this problem)

- If the system starts with no $v$ flow, then it remains in a state of no $v$ flow.
- If the system starts with no high-spatial-frequency flow components, it will remain in such a state
- One would be hard pressed to describe the exponential decay of superimposed modes described by this equation as “turbulent”

So achieving turbulence requires one of two things

- A forcing function with a $v$ component
- or an initial condition with a $v$ component.

Since our forcing function is only in $x$, we have to pick initial conditions with a $v$ flow component to get any hope of turbulence. However, the predictability of the uniform flow case makes it easy to test the numerical solver. See Section 4.


3 Numerical method

3.1 Stream function transformation

Recall that the full original equations are

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + F(y)
\]

and

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

with the constraint that

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.
\]

As mentioned in the notes, the constraint \(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\) is difficult to maintain numerically.

One way to ensure this constraint is to maintain the equivalent constraint of \(\mathbf{r}_e = u \mathbf{e}_x + v \mathbf{e}_y\) for some scalar function \(\Psi\) and \(e_x, e_y, e_z\) as the canonical basis of \(\mathbb{R}^3\). Recall that for any function, \(f\), in \(\mathbb{R}^3 \rightarrow \mathbb{R}^3\), \(\nabla \times f = 0\). By maintaining the constraint that \(\nabla \times \Psi e_z = u e_x + v e_y\), we are maintaining that \(\nabla (ue_x + ve_y) = 0\), or equivalently \(\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0\).

Such a function, \(\Psi\), is called a “stream function” and is a standard device in uid dynamics (see (1)). It has the following properties

- Along a curve \(c\) with differential tangent \(dl\) and differential normal \(dn\), the flux through the line is \(\int_c <u, v> \cdot dn = \int_c <\frac{\partial}{\partial y} \Psi, -\frac{\partial}{\partial x} \Psi > \cdot dl = \Psi_{final} - \Psi_{initial}\), making the difference between the stream function at two points the volume of flux through a curve connecting the points.

- Since streamlines are tangent to the flow, the value of \(\Psi\) must be the same along a streamline. This is very helpful for visualization.

With the addition of the stream function, the original equations become

\[
\Psi_{yt} + \Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} = -\frac{\partial \rho}{\partial x} + \nu \left( \frac{\partial^2 \Psi_y}{\partial x^2} + \frac{\partial^2 \Psi_y}{\partial y^2} \right) + F(y)
\]

and

\[
-\Psi_{xt} + \Psi_y \Psi_{xx} + \Psi_x \Psi_{xy} = -\frac{\partial \rho}{\partial y} - \nu \left( \frac{\partial^2 \Psi_x}{\partial x^2} + \frac{\partial^2 \Psi_x}{\partial y^2} \right)
\]

Cancelling out the pressure is done by subtracting \(\frac{\partial}{\partial y}\) eq1 \(-\frac{\partial}{\partial x}\) eq2. When the terms are cancelled out, we are left with

\[
-\Psi_{xx} - \Psi_y \Psi_{xxx} + \Psi_x \Psi_{xxy} - \Psi_y \Psi_{yy} - \Psi_x \Psi_{xyy} + \Psi_{yy} = -\nu (\Psi_{xxx} + \Psi_{xxy} + \Psi_{yyx} + \Psi_{yyy}) + \frac{\partial}{\partial y} F(y)
\]

This equation can be simplified by introducing a term called the “vorticity”, \(\omega\), which is defined such that \(\omega e_z = \nabla \times u\). Since \(u = \nabla \times (\Psi e_z)\), \(\omega e_z = e_z (-\frac{\partial^2}{\partial y^2} \Psi - \frac{\partial^2}{\partial x^2} \Psi)\).

The final equation is now

\[
\omega_t + \Psi_y \omega_x - \Psi_x \omega_y = \nu (\Psi_{xx} + \Psi_{yy}) + F(y).\]

The numerical method used is described in Table 1.

3.2 Transformed Boundary Conditions

In order to make calculations easier, we use the boundary conditions that \(v = 0\) and \(\frac{\partial \Psi}{\partial y}\) at the top and bottom boundaries.

This is convenient, as it entails \(\omega = \nabla \times (u, v, 0)\) is zero at the top and bottom boundaries. It also entails that \(\frac{\partial}{\partial x} \Psi = 0\) at the boundaries. The only remaining question is which constants to pick for \(\Psi\). It turns out that there
Name: Numerical method for Couette flow
Goal: Create a stable numerical method for simulating incompressible 2d Navier-Stokes with simple boundary conditions
Assumes: Solutions exist to the Navier Stokes equations for most initial conditions

1: Set \( \vec{u} = (u, v) \) according to the initial conditions.
   \{Generate \( \omega \) from numerically evaluating \( \nabla \times \vec{u} \}\}
2: for all \( \omega_{ij} \) do
3:   \( \omega_{ij} := \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2Y} - \frac{u_{i+1,j} - u_{i,j}}{2X} \)
4: end for
5: for all iterations \( i \) do
6:   Calculate \( \Psi \) from \( \omega \) by performing conjugate gradient to invert \( \nabla \nabla \Psi = \omega \) using the last \( \Psi \) as a guess.
7:   if \( i < \text{degree(Adams Bashforth)} \) then
8:     Calculate linear and non-linear components of \( \omega_t \)
9:   Explicitly step \( \omega \)
10:  Store non-linear component of \( \omega_t \)
11:  else
12:     Plug past non-linear terms into Adams Bashforth
13:     Call the result times \( \delta T \delta_{NL} \)
14:     Letting \( \Delta \) be the linear operator for the Laplacian, use conjugate gradient to apply inverse of \( (I + \Delta) \) in the Crank Nicholson equation \( \frac{1}{2}(I + \Delta)\omega_{\text{linear}}^{\text{next}} = \frac{1}{2}(I + \Delta)\omega_{\text{linear}}^{\text{curr}} + F \).
15:     \( \omega_{\text{next}} := \omega_{\text{linear}}^{\text{next}} + \delta_{NL}. \)
16: end if
17: if \( i \mod n_{\text{print freq}} = 0 \) then
18:   Calculate \( (u, v, 0) = \nabla \times \Psi_{ez}. \)
19:   Save out \( \Psi, u \) and \( v. \)
20: end if
21: end for
22: End algorithm.

Table 1: Overall Algorithm.
are many choices for \( \Psi \) which work. I picked \( \Psi = 0 \) It can be proven that for any choice of interior values of \( \omega \) such that \( \sum_{i,j} \omega_{ij} = 0 \) (this corresponds to a lack of net inflow or outflow of material due to Stokes’ theorem), there exists a setting of the interior values of \( \Psi \) with 0 exterior boundaries such that \( \Delta \Psi = \omega \). One simple proof is noting that \( \Delta \) is a graph Laplacian operator over a connected graph – never mind, I’ll finish this thought later.

### 3.3 Conversions

### 3.4 Term splitting

### 3.5 Numerical method

For an overview of the algorithm, see Table 1.

What follows is a listing of the various files and what they do. The files described here can be found at http://www.soe.ucsc.edu/ mds/couette along with the current copy of this report.

- **main.f** has the main program, which consists of one procedure call. This call can be swapped out to run various test functions
- **runSim.f** contains three procedures
  - runSim is the main procedure and contains the initial conditions for the print frequency, the timestep (fixed for Adams Bashforth), the aspect ratio and the grid spacing
  - DoRkStep does the initial non-implicit step to kickstart semi-implicit Adams Bashforth
  - DoSIABStep does the semi-implicit adams bashforth
- **iniVals.f** contains functions for all the other initial conditions
- **initializeFlow.f** sets the initial flow. If you want a random valid flow, replace it with the file “alt init rename to initializeFlow dot f”
- **testChain.f** and **TestPsiFromW.f** just test the conversion chain
- **derivs.f** calculates the linear and nonlinear components of the derivative of \( \omega \)
- **multLat2d.f** applies the linear operator \( \Delta \) to a given \( \Psi \)
- **conjgrad.f** contains two conjugate gradient routines, one which operates over linear functions applied to 1d vectors (not currently used) and one which operates over linear functions applied to 2d arrays
- **psiFromW.f**, **uFromPsi.f** and **wFromU.f** are conversion routines
- **printresult.f** contains the print functions I need
- **arrayManip.f** contains array manipulation primitives to allow me to write code that looks more like matlab code (or Fortran90 from what I have heard)

### 4 Testing

For the test, I picked the forcing function corresponding to \( u_{initial}(x, y) = -\nu \frac{\partial^2}{\partial y^2} F(y) \) and the forcing function as described in class, and checked that I had started on a fixed point.

Then I repeated the experiment with a different initial condition that also depended solely on \( x \) and checked that the system converged to \( -\nu \frac{\partial^2}{\partial y^2} \).
4.1 Numerical Diffusion

One way to test the numerical diffusion of the non-linear component would be to set viscosity to zero, change the boundary conditions to fully toroidal, initialize the flow to be 0 outside of a given non-axis-aligned channel and uniform in the channel direction inside the channel, run for one time step, and compare the Fourier transform of the initial condition and the result.

5 Turbulence

5.1 Discussion of $v = 0$ case

5.2 Choice of initial conditions

Luckily we can pick a consistent initial flow if we pick some arbitrary function $\Psi$ that satisfies $\Psi$’s boundary conditions, and set the initial condition for $u$ to be its curl.

I picked $\Psi e_z = \sin(2\pi x) \sin(2\pi y) e_z$, which is zero at the boundaries and has a curl of $< -2\pi \sin(2\pi x) \sin(2\pi y), 2\pi \cos(2\pi x), 0 >$.

6 Visualization

In general, picking seed points from which to draw streamlines is a hard problem in visualization. The nice thing about incompressible flow in general, and the method we are using in particular, is that surfaces of constant $\Psi$ correspond to streamlines.

In order to visualize streamlines (see Figure 3) I found the min and max values of $\Psi$, and used the square of sin function with a frequency of 20 times this difference to map the $\Psi$ values onto the red channel of color.

On top of this I drew blew glyphs corresponding to (scaled) flow magnitude and direction. The ellipses on the glyphs are actually the tails of the vectors rather then the heads.

![Figure 3: Single frame of Couette flow](image)

7 Conclusions and future work

References